SISTEMI LINEARI E STAZIONARI A TEMPO CONTINUO

Movimento ed equilibrio

Stabilità

MOVIMENTO ED EQUILIBRIO

- Sistema lineare e stazionario ($u \in R^m, x \in R^n, y \in R^p$)

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Formula di Lagrange

- In risposta a $u(t), t \geq t_0, x(t_0) = x_{t_0}$:

$$x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$
$$y(t) = Ce^{A(t-t_0)}x_{t_0} + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

⋆ movimenti liberi ($t_0 = 0$)

$$x_l(t) = e^{A(t-t_0)}x_{t_0}$$
$$y_l(t) = Ce^{A(t-t_0)}x_{t_0}$$

⋆ movimenti forzati

$$x_f(t) = \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$
$$y_f(t) = C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Rappresentazioni equivalenti

- Trasformazione di stato

$$\hat{x}(t) = Tx(t)$$
$$x(t) = T^{-1}\hat{x}(t)$$

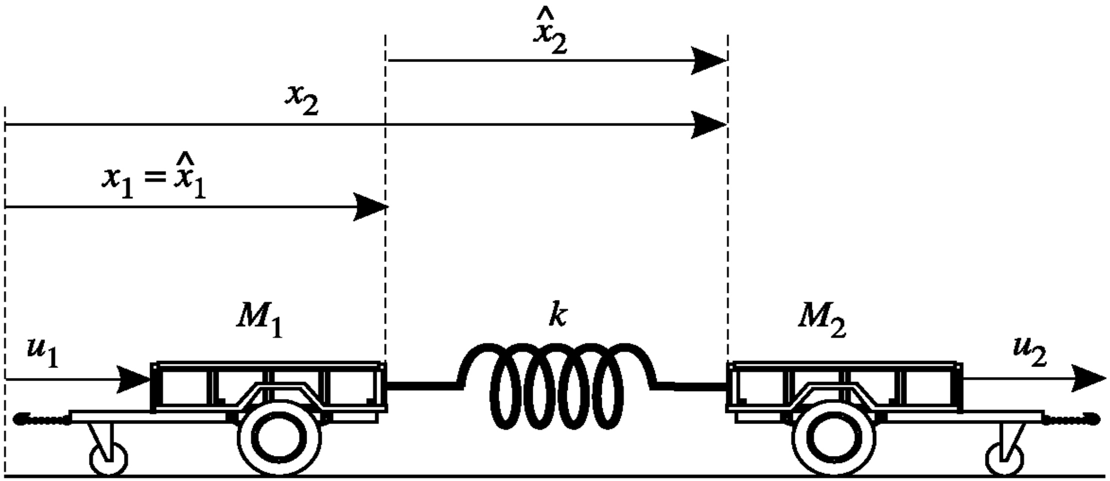
$$\Downarrow$$

$\star$ sistema dinamico equivalente (A e $\hat{A}$: stessi autovalori)

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t)$$
$$y(t) = \hat{C}\hat{x}(t) + \hat{D}u(t)$$

$$\hat{A} = TAT^{-1} \quad \hat{B} = TB \quad \hat{C} = CT^{-1} \quad \hat{D} = D$$

- Esempio: sistema meccanico



$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{M_1} & \frac{1}{M_1} & 0 & 0 \\ \frac{k}{M_2} & -\frac{k}{M_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{M_1} & 0 \\ 0 & \frac{1}{M_2} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

⋆ trasformazione di stato

$$
\begin{aligned}
\hat{x}_1(t) &= x_1(t) \\
\hat{x}_2(t) &= x_2(t) - x_1(t) \\
\hat{x}_3(t) &= x_3(t) \\
\hat{x}_4(t) &= x_4(t) - x_3(t)
\end{aligned}
$$

$$
T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}
$$

$$\Downarrow$$

$$
\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \\ \dot{\hat{x}}_3(t) \\ \dot{\hat{x}}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{k}{M_1} & 0 & 0 \\ 0 & -\frac{(M_1+M_2)k}{M_1 M_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \\ \hat{x}_4(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{M_1} & 0 \\ -\frac{1}{M_1} & \frac{1}{M_2} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}
$$

$$
y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \\ \hat{x}_4(t) \end{bmatrix}
$$

Autovalori e modi

- Matrice della dinamica diagonalizzabile (autovalori distinti: $T = T_D$)

$$\hat{A} = \hat{A}_D = \mathrm{diag}\{s_1, s_2, \ldots, s_n\}$$

  ⋆ movimento libero dello stato

$$\begin{aligned}
\hat{x}_l(t) &= e^{\hat{A}_D t}\hat{x}_0 = \sum_{k=0}^{\infty} \frac{\left(\hat{A}_D t\right)^k}{k!}\hat{x}_0 \\
&= \mathrm{diag}\left\{\sum_{k=0}^{\infty}\frac{(s_1 t)^k}{k!}, \sum_{k=0}^{\infty}\frac{(s_2 t)^k}{k!}, \ldots, \sum_{k=0}^{\infty}\frac{(s_n t)^k}{k!}\right\}\hat{x}_0 \\
&= \mathrm{diag}\{e^{s_1 t}, e^{s_2 t}, \ldots, e^{s_n t}\}\hat{x}_0
\end{aligned}$$

$$\Downarrow$$

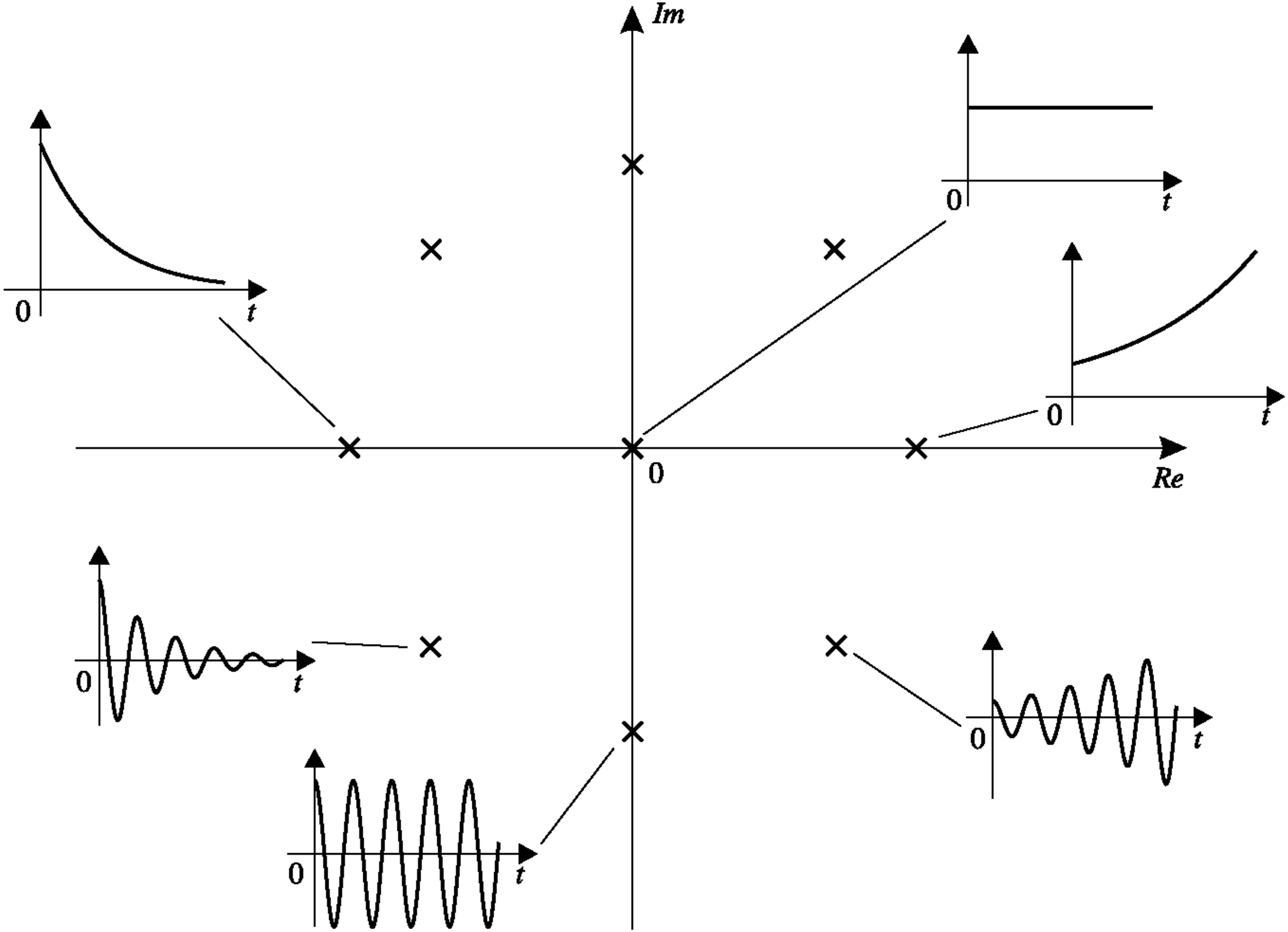
$$\begin{aligned}
x_l(t) &= T_D^{-1}\hat{x}_l(t) = T_D^{-1}\mathrm{diag}\{e^{s_1 t}, e^{s_2 t}, \ldots, e^{s_n t}\}T_D x_0 \\
y_l(t) &= C T_D^{-1}\mathrm{diag}\{e^{s_1 t}, e^{s_2 t}, \ldots, e^{s_n t}\}T_D x_0
\end{aligned}$$

$$x_l(t) = T_D^{-1}\hat{x}_l(t) = T_D^{-1}\mathrm{diag}\{e^{s_1 t}, e^{s_2 t}, \ldots, e^{s_n t}\}T_D x_0$$

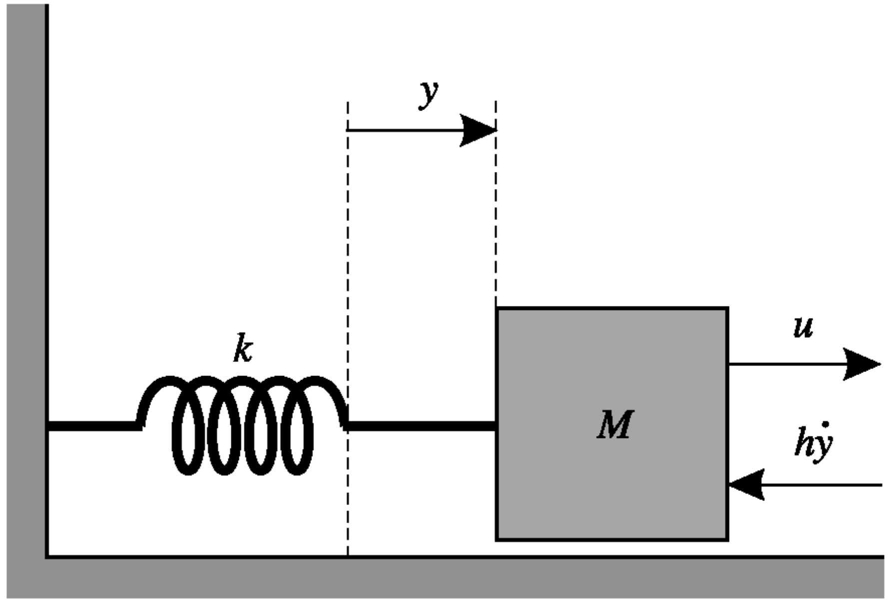
$$y_l(t) = CT_D^{-1}\mathrm{diag}\{e^{s_1 t}, e^{s_2 t}, \ldots, e^{s_n t}\}T_D x_0$$

- $\star$ $e^{s_i t}$: modi aperiodici
- $\star$ $e^{\sigma_i t}\sin(\omega_i t + \varphi_i)$: modi pseudoperiodici

- Modi dei sistemi dinamici con autovalori distinti



- Esempio: sistema massa-molla



$\star$ legge fondamentale della dinamica

$$M\ddot{y}(t) = -ky(t) - h\dot{y}(t) + u(t)$$

⋆ legge fondamentale della dinamica

$$M\ddot{y}(t) = -ky(t) - h\dot{y}(t) + u(t)$$

⋆ rappresentazione di stato ($x_1 = y$, $x_2 = \dot{y}$)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/M & -h/M \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

⋆ autovalori

$$s_1 = -\frac{h}{2M} + \sqrt{\frac{h^2}{4M^2} - \frac{k}{M}} \qquad s_2 = -\frac{h}{2M} - \sqrt{\frac{h^2}{4M^2} - \frac{k}{M}}$$

$\star$ autovalori distinti ($h^2 \neq 4MK$)

$$T_D = \frac{1}{s_2 - s_1} \begin{bmatrix} s_2 & -1 \\ -s_1 & 1 \end{bmatrix}$$

$$\hat{A}_D = T_D A T_D^{-1} = \text{diag}\{s_1, s_2\}$$

$$\Downarrow$$

$\star$ movimenti liberi ($h^2 > 4MK$)

$$x_l(t) = \frac{1}{s_2 - s_1} \begin{bmatrix} (s_2 x_{01} - x_{02})e^{s_1 t} - (s_1 x_{01} - x_{02})e^{s_2 t} \\ (s_1 s_2 x_{01} - s_1 x_{02})e^{s_1 t} - (s_1 x_{01} - x_{02})e^{s_2 t} \end{bmatrix}$$

$$y_l(t) = \frac{1}{s_2 - s_1} (s_2 x_{01} - x_{02})e^{s_1 t} - (s_1 x_{01} - x_{02})e^{s_2 t}$$

$\star$ movimenti liberi ($h^2 < 4MK$)

$$\sigma = -\frac{h}{2M} \qquad \omega = \sqrt{\frac{k}{M} - \frac{h^2}{4M^2}}$$

$$\delta = \sqrt{\sigma^2 + \omega^2} \qquad \gamma = \arcsin\frac{\omega}{\delta} = \arccos\frac{\sigma}{\delta}$$

$$x_l(t) = e^{\sigma t} \begin{bmatrix} -\frac{\delta}{\omega}\sin(\omega t - \gamma)x_{01} + \frac{1}{\omega}\sin(\omega t)x_{02} \\ -\frac{\delta^2}{\omega}\sin(\omega t)x_{01} + \frac{\delta}{\omega}\sin(\omega t + \gamma)x_{02} \end{bmatrix}$$

$$y_l(t) = e^{\sigma t}\left(-\frac{\delta}{\omega}\sin(\omega t - \gamma)x_{01} + \frac{1}{\omega}\sin(\omega t)x_{02}\right)$$