TIME SERIES ANALYSIS

Modelling : understanding of the physical system to write dynamical equations

⇓

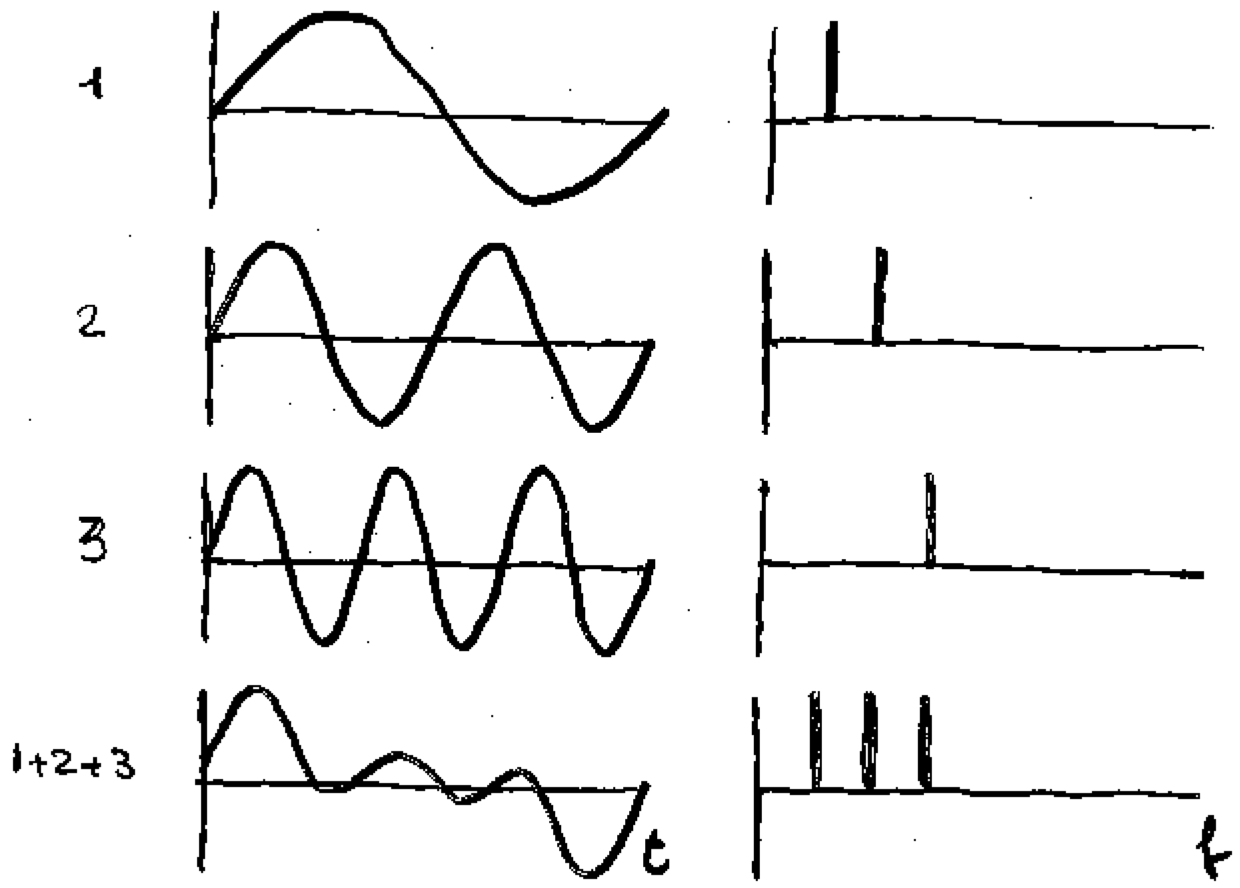
Here we take the opposite approach

Starting with a sequence of measurements - a TIME SERIES - we want to see what the data themselves can tell us about the dynamics.

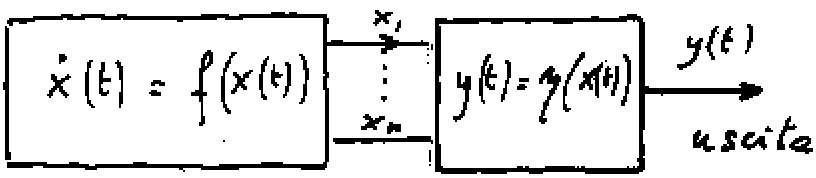
ULTIMATE GOAL : to construct a computer program that, without any knowledge of the physical system from which the data come, can take the data as input and provide as output a mathematical model

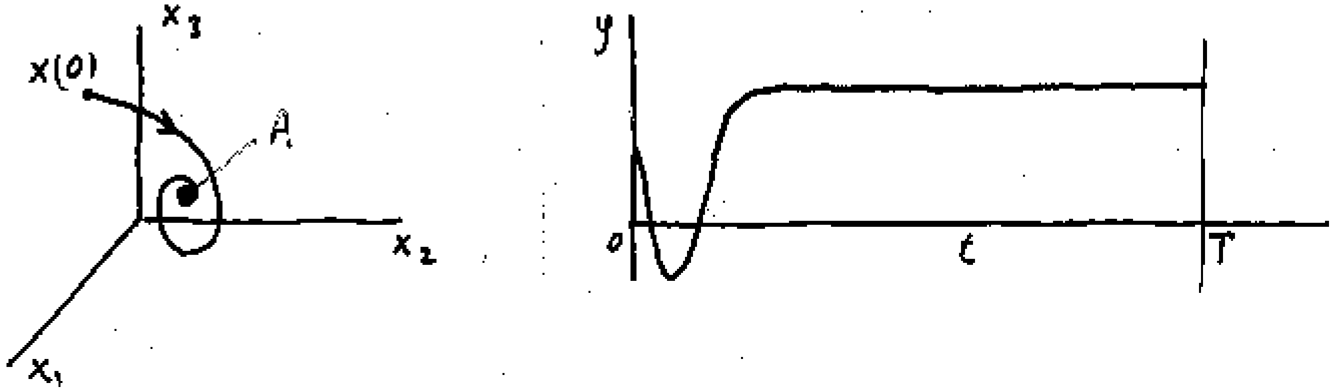
3 single frequency sinusoids and their summation

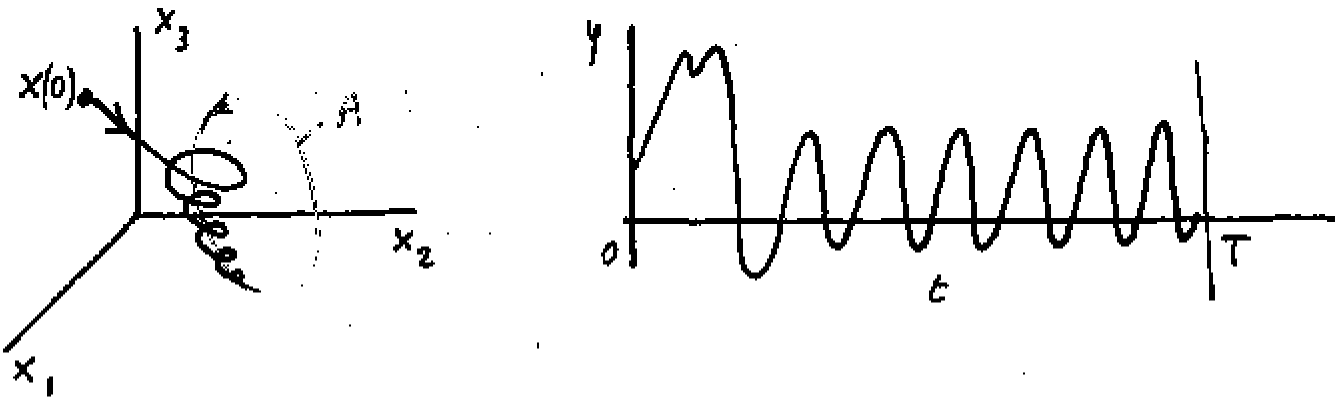
FFT

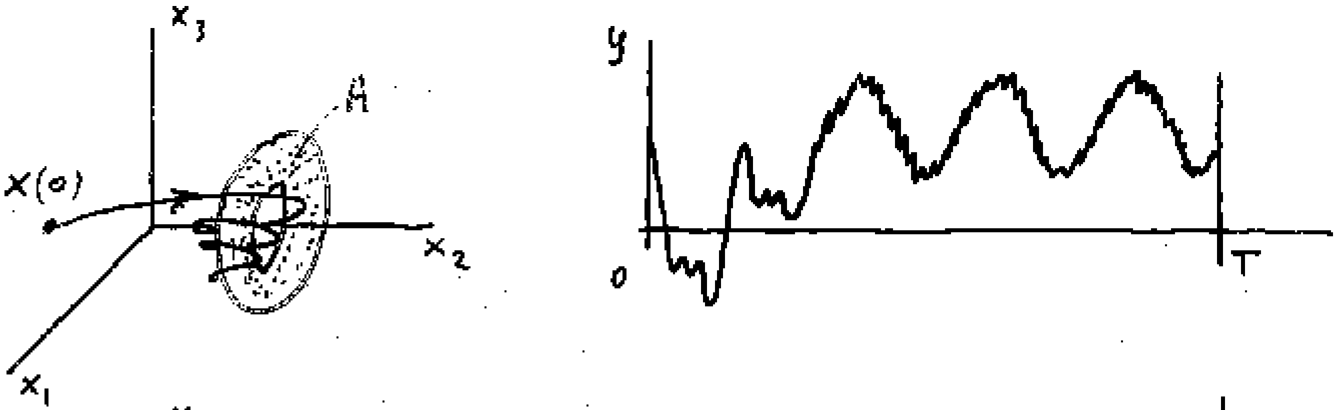


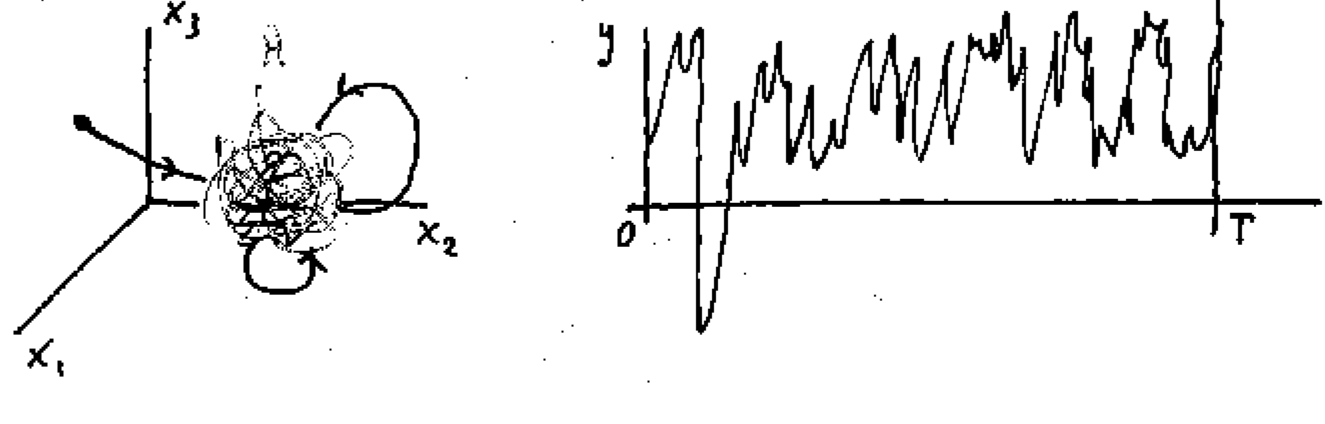
SERIE TEMPORALI











STRANGE ATTRACTOR

It is NOT : A TORUS
A CYCLE or
AN EQUILIBRIUM

PROPERTIES :

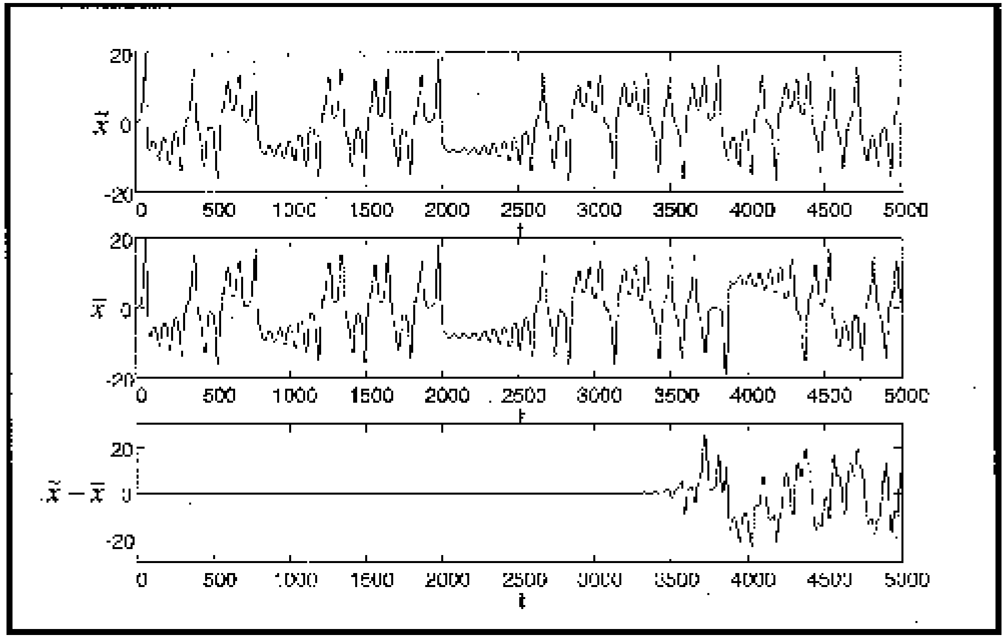
1 NON INTEGER DIMENSION
2 DIVERGENT TRAJECTORIES

Point 1 ⟶ FRACTALITY

Point 2 ⟶ SENSITIVITY TO STARTING CONDITIONS

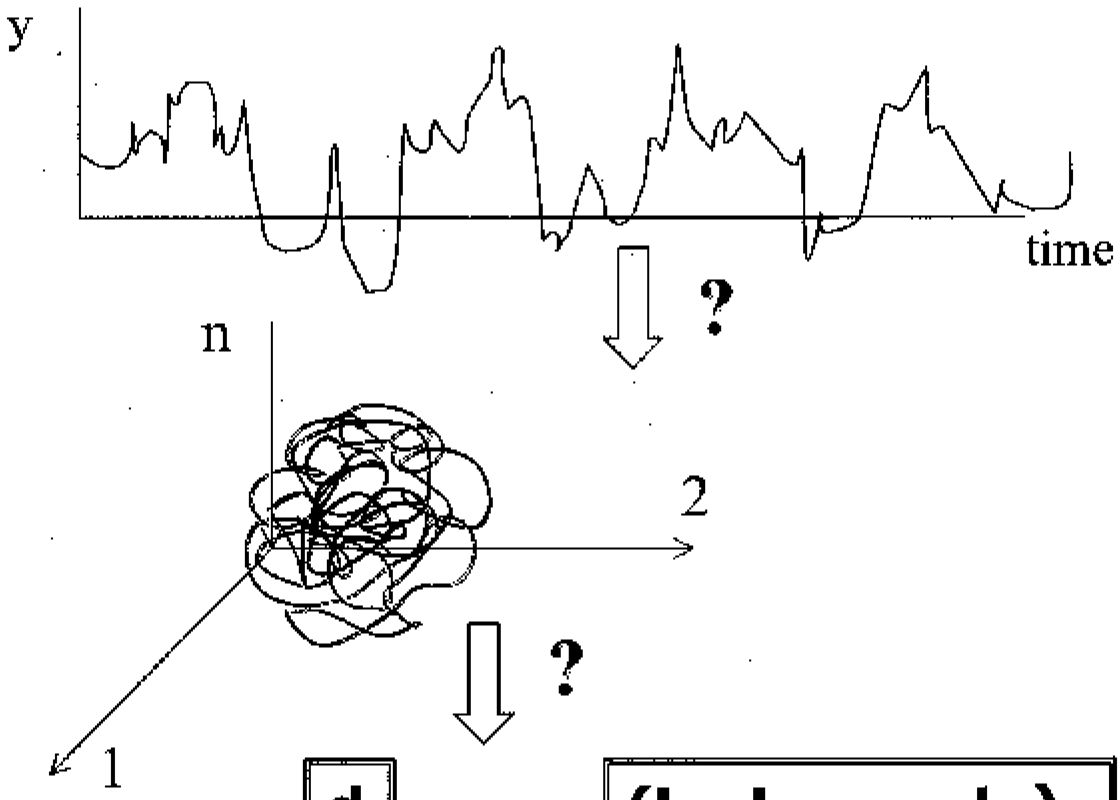
Local divergence is compensated by the global folding of trajectories (stretching & folding)

Chaotic Systems



Properties:

- Determinism
- Aperiodicity and bounded dynamics
- Presence of *Strange Attractors* with fractal (non integer) dimension
- Entropy K_2 convergent
- Sensitive dependence to initial conditions (Lyapunov Exponents)



$\boxed{\mathbf{d}}$ $\boxed{(\mathbf{L_1, L_2, \ldots, L_n})}$

If the time series has been recorded when the system is on the attractor we correctly estimate only

$(L_1, L_2, \ldots, L_d{}^+)$ exponents

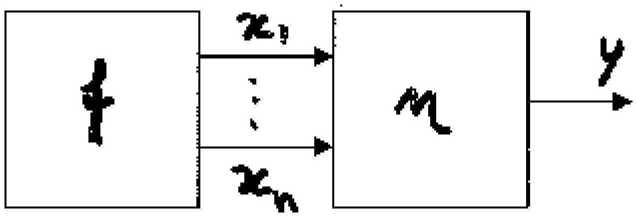
- **Recall**

$$d = d^- + \frac{L_1 + L_2 + \ldots + L_{d^-}}{|L_{d+}|}$$

- (KY conjecture)

- We will first assume that the system is **known**

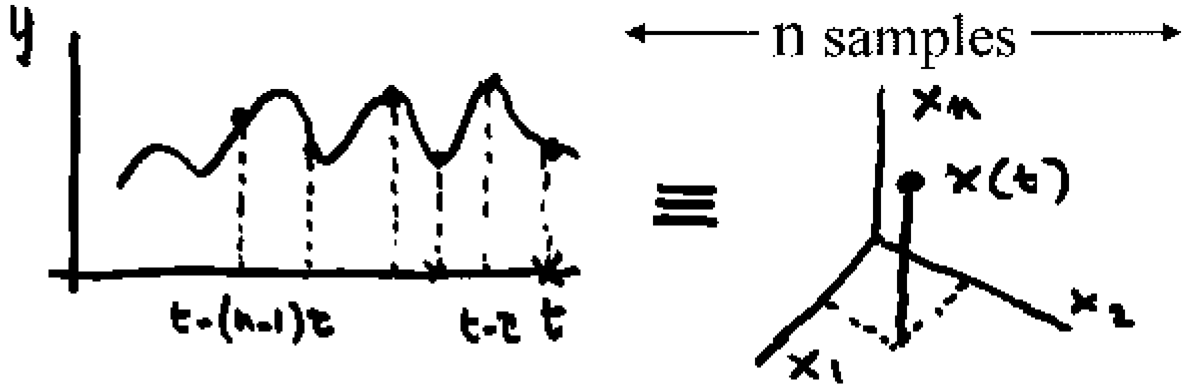
State Reconstruction



$$\underbrace{y_{[0,t]},\ f,\ \eta}_{\Downarrow\ ?\ x(t)}$$

Takens(1981): $\forall\ f, \eta, \tau$

$\boxed{x(t) \text{ is "}\textbf{equivalent}\text{" to } (y(t), y(t-\tau), \ldots, y(t-(n-1)\tau}$



- Notice that Takens statement does not say how **x(t)** can be computed from the **n** samples.

For the case of linear systems This is know since 1960. (Kalman) .

Moreover in the linear case **x(t)** can be easily computed from **n** successive samples of the output sequence (They are equivalent to the ***n*** components of the state vector).

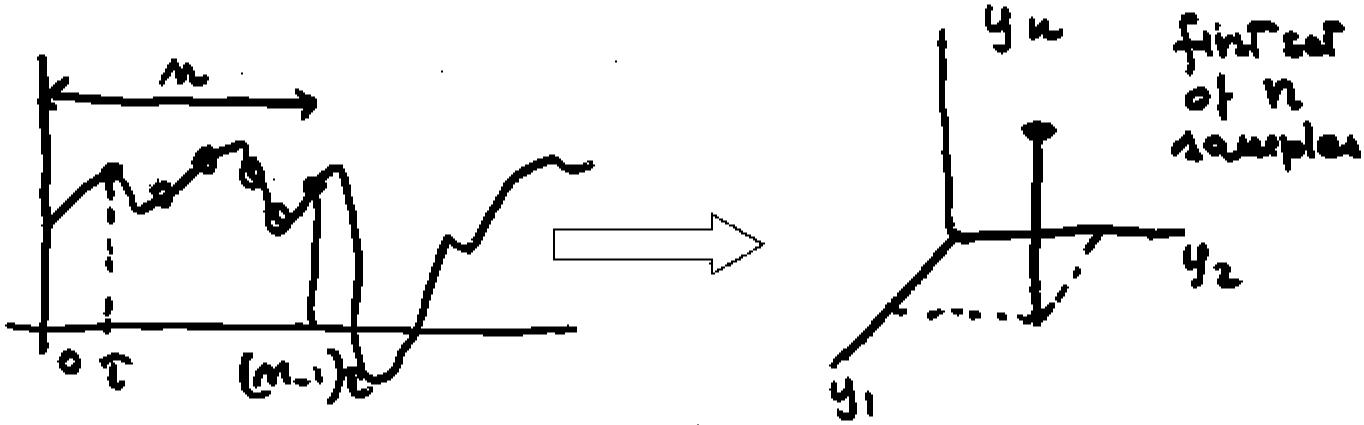
Embedding
(reconstruction in a space with dimension **n**)

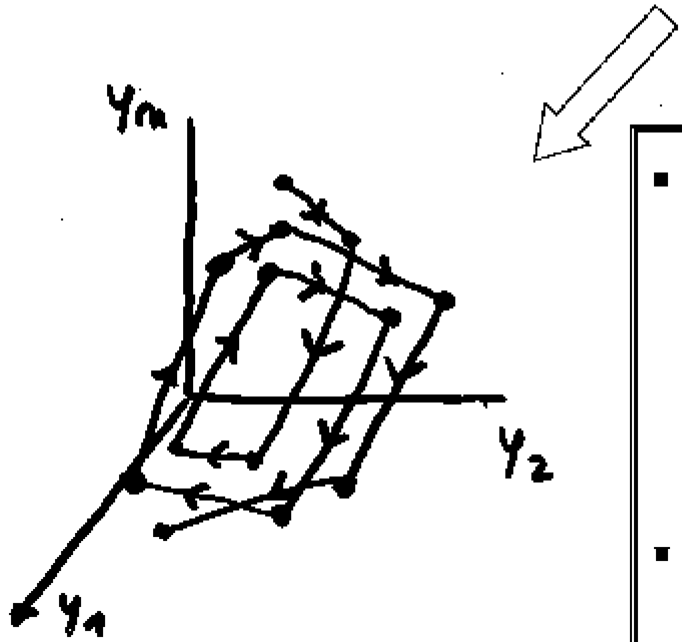
- The technique of representing a measured time series as a sequence of points in a n-dimensional space is called **time-lag embedding**
- Takens embedding theorem (1981) says that:

  the reconstructed dynamics are **geometrically similar** to the original for both continuous-time and discrete-time systems.

- $(y(t), y(t-\tau), \ldots, y(t-(n-1)\tau$:

  The sequence of points created by embedding a time series is the **trajectory**. **n** is called embedding dimension and τ is the embedding lag





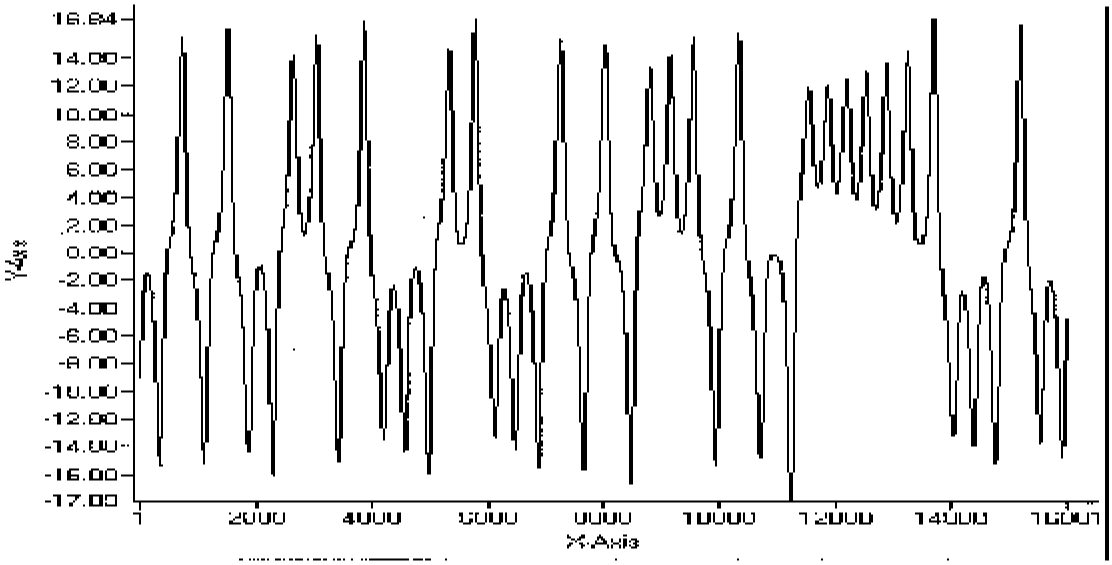
- Reconstructed attractor has the same dimension and the same Lyapunov exponents than the true attractor.
- They are invariant parameters

It is important to properly select τ

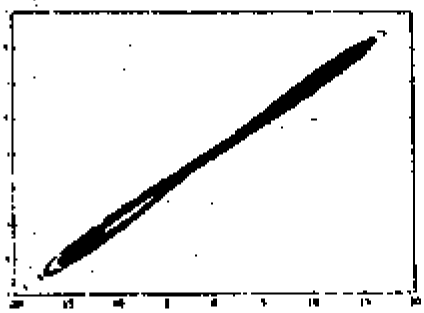
Theoretically, all τ intervals are acceptable.

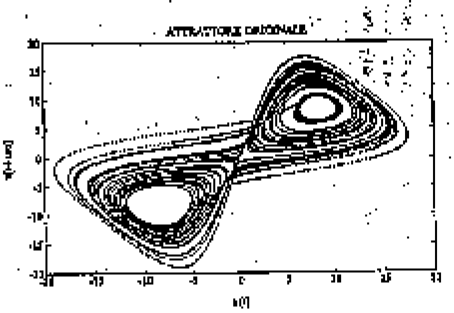
In practice, only a quite restricted range of τ values allow to correctly reconstruct the attractor.

L'influenza di τ

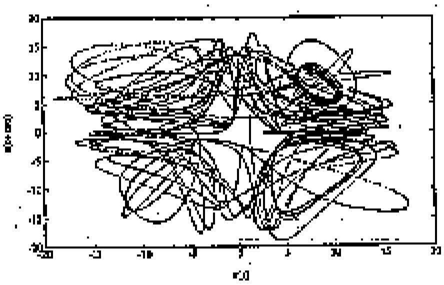


$\tau = 5$





$\tau = 54$

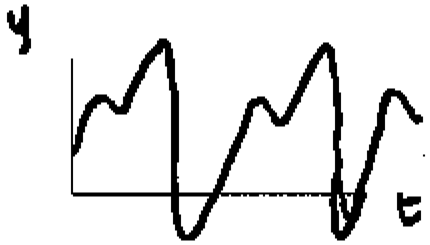


$\tau = 915$

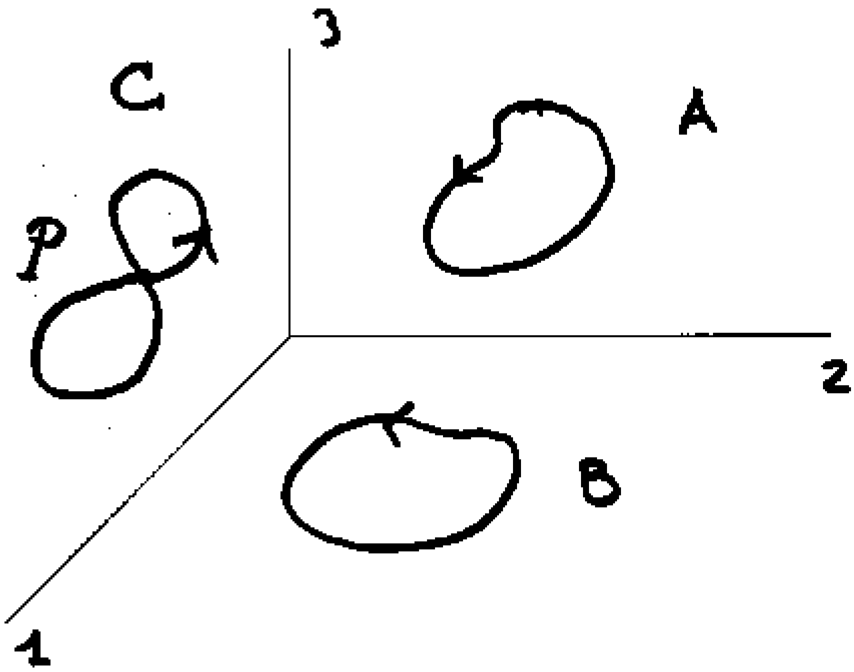
- 3 examples of the reconstruction of the Lorenz attractor from its variable z
(τ= 8, 51, 915)

Reconstruction dimension m

Assume the system is 3-dimensional and that the attractor is a cycle (**n**=3, **d**=1)

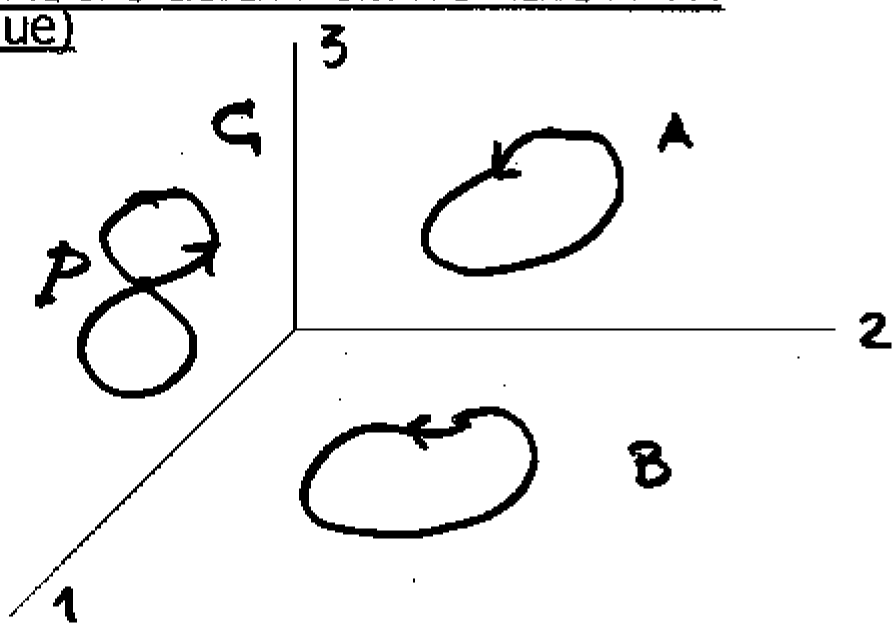


If we are **on the attractor** the time series is periodic



- Case A: reconstruction is made with m=3 delayed values of the output.
- Case B e C with m=2 delayed values of the output.

Reconstruction dimension m
(continue)

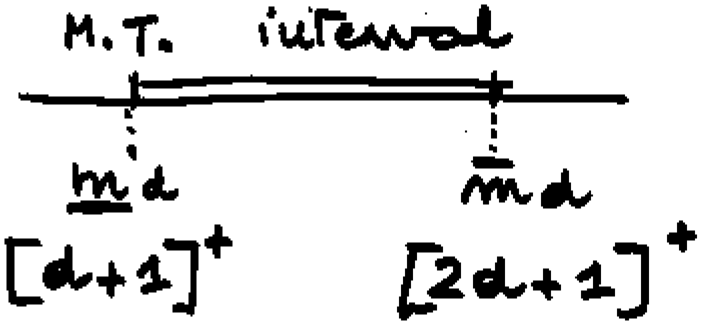


- If we use **m=3** delayed samples, we will reconstruct the cycle A in a three-dimensional space.
- If we use **m=2** delayed samples, we will reconstruct the cycle B or C in a two-dimensional space.
- In case C there are critical points P : the pointson the cycle C near to point P look close one to each other but they can be far in the true state space.
- They are False Nearest Neighbors
- Any algorithm will give d=1 in all cases.
- Only the first exponent (L_1=0) can be computed from these data. The exponent is correctly calculated in case A (m=3) and in case B (m=2) but in case C (m=2) the algorithm may fail

Extension

	Cycle (d=1)	Torus (d=2)	Strange Attractor (d >2)
for estimating the dimension d	m=2	m=3	$m = [d+1]^+$
			Manè-Takens
for estimating the first d^+ Lyapunov exponents	$2 \le m \le 3$	$3 \le m \le 5$	$[d+1]^+ \le m \le [2d+1]^+$

Obviously, we must also have $\mathbf{m \le n}$

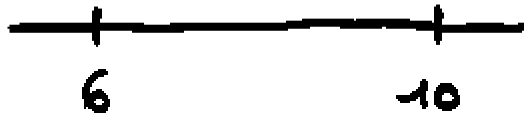


- With $m=\underline{m}_d$
  - we will estimate the dimension and, in lucky cases, even the d^+ Lyapunov Exponents

Example

- *Cardiac rhythm (ECG)*
  - n=? ($n \geq 10$ because the system is very complex)
  - d=4.1

$$[4.1+1]^{+} \qquad [8.2+1]^{+}$$



  - In the most lucky case the Lyapunov exponents $L_1, L_2, \ldots, L_n$ can be computed with m=6 delayed samples
- *Lorenz System*
  - n=3, d=2,0...

$$\underline{m}_d = [2.0\ldots+1]^{+} = 4$$
$$\bar{m}_d = [4,\ldots+1]^{+} = 6$$

  - But **m** must be smaller than or equal to **n**
  - **m=3** is good in this case

n is large (generally unknown)